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Two Remarks on the Decreasing Rearrangement of a Function

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We show that an almost everywhere differentiable function is singular iff its decreasing rearrangement is singular. Also we give a remark on a theorem of Duff. © 1987 Academic Press, Inc.

1. A CHARACTERIZATION OF SINGULAR FUNCTIONS

By a well-known theorem of Banach and Zaretzki a continuous function of bounded variation is absolutely continuous iff it is null set preserving. There is an analogous characterization of singular functions which is less known (see Theorem 4.14 of [1]).

PROPOSITION 1. *An almost everywhere differentiable function $f\colon (0, 1) \to \mathbb{R}$ is singular (that means $f' = 0$ a.e.) iff there is a set A in $(0, 1)$ with full measure such that $f(A)$ is a null set.*

2. APPLICATION TO THE DECREASING REARRANGEMENT

If $f\colon (0, 1) \to \mathbb{R}$ is measurable we define the decreasing rearrangement $f^*\colon (0, 1) \to \mathbb{R}$ by $f^*(x) = \sup\{y \mid \lambda\{t \mid f(t) > y\} > x\}$.

LEMMA. *For a measurable $f\colon (0, 1) \to \mathbb{R}$ there exists an A in $(0, 1)$ with $\lambda(A) = 1$ and $\lambda(f(A)) = 0$ iff such a set A exists for f^*.*

Proof. Let f be measurable and $\lambda(A) = 1$, $\lambda(f(A)) = 0$. Then $f(A)$ is contained in some Borel measurable null set B. Because of $\lambda(f^{-1}(M)) = \lambda(f^{*-1}(M))$ for arbitrary Borel sets M, it follows that $\lambda(f^{*-1}(B)) = 1$. The other direction follows in the same way.

PROPOSITION 2. *An almost everywhere differentiable $f\colon (0, 1) \to \mathbb{R}$ is singular iff its rearrangement f^* is singular.*

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Remark. This follows easily from Proposition 1 and the lemma. Proposition 2 shows that a remark in [2] is not correct. In this paper it is stated that there is a continuous function of bounded variation with $f' \neq 0$ on a set of measure $1-\varepsilon$ and $f^{*\prime} = 0$ a.e.

3. A Remark on a Theorem of Duff

In the paper [3] the following is stated as Theorem 1:

(*) Let $f: (0, 1) \to \mathbb{R}$ be differentiable a.e. and let $G: \mathbb{R}^+ \to \mathbb{R}$ be a convex function.

Then

$$\int_0^1 G(|f^{*\prime}(x)|)\, dx \leqslant \int_0^1 G(|f'(x)|/n(f(x)))\, dx.$$

Here $n(y)$ is the number of roots of the equation $y = f(x)$ in $(0, 1)$. We indicate that this important inequality is not valid in such a general form.

Examples. (1) Let C be the Cantor set in $(0, 1)$ and let $f: (0, 1) \to (0, 1)$ be a function such that $f(x) = x$ for all $x \notin C$ and $n(y) = 2$ for all $y \in (0, 1)$. Then f is differentiable a.e. with

$$\int |f^{*\prime}(x)|\, dx = 1, \qquad \text{but} \quad \int |f'(x)|/n(f(x))\, dx = \tfrac{1}{2}.$$

(2) Let $f: (0, 1) \to \mathbb{R}$ be continuous, linear, and isotone on each interval $I_n = (1/(n+1), 1/n)$ $(n \in \mathbb{N})$ with $f(I_n) = (0, 1)$ for all n. Then $f^* = 1 - \mathrm{id}$, $\int |f^{*\prime}(x)|\, dx = 1$ but $\int |f'(x)|/n(f(x))\, dx = 0$.

Remarks. (a) These examples may be excluded by the demand that $n(y) = \# \{x \mid f(x) = y \text{ and } f'(x) \text{ exists}\}$ and that $n(f(x))$ is finite a.e.

(b) The proof of (*) given in [3] uses an equality which is so far apparently proved only for continuously differentiable functions (see [4, Lemma 1]).

References

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